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Mean-field renormalization group approach to two-dimensional site directed percolation

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Abstract. Two-dimensional site directed percolation is studied by the mean-field renormalization group approach to bulk and surface critical properties Extrapolation techniques for the exponents and the percolation threshold are described A very good estimate of the percolation threshold ($p_c = 0\,7055 \pm 0\,0001$) and the first evaluation of the surface exponent $y_{hs} = 0\,653 \pm 1$ are obtained

1. Introduction

It is well known that mean-field-like approximations always give classical critical exponents. However, mean-field approaches can be improved by the introduction of scaling concepts in such a way that non-classical exponents can be obtained.

The first of these methods has been introduced by Indekeu, Maritan and Stella [1], namely the mean-field renormalization group (MFRG). These authors have established an important bridge between the classical and modern theories of critical phenomena. Using two small clusters and scaling appropriately the effective field on the boundary of the cluster they have obtained a renormalization group mapping The fixed point and the associated critical exponents have been good estimates of the real critical parameters. This method has been applied to a wide range of critical phenomena like classical and quantum spin models, both ordered and disordered, dynamical critical phenomena in Glauber model (see [2] and references therein), geometrical critical phenomena and percolation [3].

In order to improve these ideas and consider surface effects, Indekeu *et al* [2] have proposed a new MFRG with an unified approach to bulk and surface critical behaviours. This approach has improved the accuracy of the previous MFRG results because the boundary effective field must be scaled as a surface field Now, the renormalization group transformation is obtained by considering three clusters, instead of two clusters. We will denote this new version of the method as the *three*-cluster MFRG and the previous version as the *two*-cluster MFRG.

Some of the reasons for the success of MFRG is the possibility of easy and wide applications at very low computational cost. In this paper we apply the three-cluster MFRG to the site directed percolation problem [4] in the square lattice. Firstly, we evaluate exactly the percolation probability of finite clusters in the spirit of a mean field. Afterwards we interpret this physical quantity following the three clusters MFRG. The directed percolation problem has been chosen because one can use a personal computer to consider relatively big clusters, in such way that we can investigate the convergence of the MFRG results. Moreover, it is the first time that this problem has been studied with the three-cluster MFRG. It is worth mentioning that the site and bond

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two-dimensional directed percolation problems have been previously considered in the framework of the two-cluster MFRG [5] with large clusters. The results, however, have not been so good. In particular, the critical threshold was slightly out from the expected results from series expansions. This fact has also motivated the use of the three-cluster MFRG in our work, since it is expected that the appropriate scaling of the effective field, which must scale as a surface field, may improve the results. Moreover, we discuss the extrapolation processes of the percolation threshold and the exponents for the three-cluster MFRG. These processes are not evident because we need three clusters (and not two clusters, as usual) to establish the renormalization mapping.

This paper is organized as follows. In section 2, we present the sequence of mean-field clusters as well as the method of evaluating the main quantity which will be used later on. Section 3 is concerned with the interpretation of the results obtained in the previous section in the framework of the three-cluster MFRG. In the last section we present our summary.

2. Mean-field clusters

In site directed percolation [4], a site of the lattice is occupied with a probability p and is unoccupied with a probability $1-p$, independently of the other sites. The bonds of the lattice have an arrow directed towards a privileged direction, in such a way that a cluster of occupied sites begins in an origin site and extends following the privileged direction (the vertical direction in figure 1). The order parameter is given by $P(p)$, the probability that the site at the origin is a source of an infinite cluster. The 'external field' is described by h, the probability that a given site is connected to an 'outside ghost site' by a bond [6]. The percolation probability can be written in terms of h as

$$P(p,h)=1-\frac{\partial f}{\partial h}=1+\frac{(1-h)}{p}\frac{\partial}{\partial h}\sum_s n_s(p)(1-h)^s. \tag{2.1}$$

Here $sn_s(p)$ is the probability that the origin is the source of a cluster of connected s sites, the sum over s is restricted to finite clusters and f is the equivalent of the free energy per site [7].

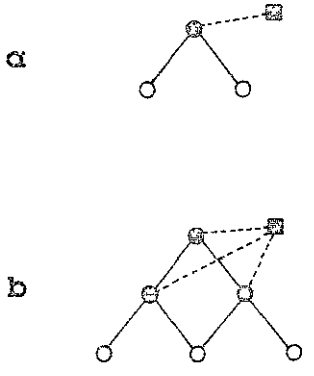

Figure 1. The two smallest mean-field clusters Empty circles represent the mean-field sites The dashed lines are the bonds connecting the sites of the cluster to the ghost site (represented by a square). (*a*) A cluster with $N=1$ site (*b*) $N=3$

The mean-field calculations follow the work of De Bell [3]. We evaluate $P(p, h)$ exactly for a cluster of N sites. We consider asymmetric clusters with $N=1$ up to $N=78$ sites. Two of these clusters are depicted in figure 1. This procedure is equivalent to considering symmetric clusters with the middle site being the origin of the directed percolation problem. Let b be the probability that a site adjacent to the cluster (a mean-field site) is connected to the infinite cluster. The bond connecting the origin to the ghost site is present with probability h. If this is the case we have a percolating configuration which is independent of the sites and mean-field sites. Then all configurations with the origin connected to the ghost site give us a term h in the percolation probability. When this bond is absent, we must consider configurations that percolate by the mean-field sites. For example, for the $N=1$ cluster (figure 1(a)) the origin site must be occupied and at least one mean-field site must be connected to the percolation cluster. These configurations give us a term $(1-h)\{pb^2+2pb(1-b)\}$ in the percolation probability. Therefore for the $N=1$ cluster we have

$$P(p, h, b)=h(1-2pb)+p(-b^2+2b) \tag{2.2}$$

up to terms of order of hb and b^2. The mean-field approximation is obtained by setting $b=P(p, h)$. This approach gives us the classical exponent $\beta=1$ and an estimate of the critical threshold $p_c=0.5$.

We use the 'transfer matrix' method of Blease [8] to evaluate exactly the coefficients of $t'_N(p, h, b)$ for the other clusters.

3. The mean-field renormalization group approach to bulk and surface critical behaviours

3.1. Definition of the rescaling transformations

In the strategy of the MFRG approach to bulk and surface critical behaviours we consider that the percolation probability of three clusters with N, N' and N'' sites in the two-dimensional case are related by the following relation

$$\begin{aligned} P_{N'}(p', h', b') &= L_{N',N}^{2-y_h} P_N(p, h, b) \\ P_{N''}(p'', h'', b'') &= L_{N'',N}^{2-y_h} P_N(p, h, b) \end{aligned} \tag{3.1}$$

where $2-y_h$ is the scaling dimension of the percolation probability and $L_{N',N}$ is the length-rescaling factor associated with clusters of N' and N sites, namely

$$L_{N',N}=\left(\frac{N}{N'}\right)^{1/2}. \tag{3.2}$$

It is important to note that one requires (3.1) to hold to leading orders in b and h. As b concerns the percolation probability of boundary sites, for large clusters it must scale like a surface field, i.e.

$$\begin{aligned} b' &= L_{N',N}^{y_{hs}} b \\ b'' &= L_{N'',N}^{y_{hs}} b. \end{aligned} \tag{3.3}$$

Here y_{hs} is the exponent related to the surface 'external field'. We determine the exponent $2-y_h-y_{hs}$ self-consistently by imposing that the two different maps of (3.1)

have the same fixed point. In this fashion the fixed point p^* and $2-y_h-y_{hs}$ are uniquely fixed. The exponents y_p and y_h are obtained from

$$\left.\frac{dp'}{dp}\right|_{p^*} = \lambda_p(p^*) = L_{N',N}^{y_p}$$
$$\lambda_h(p^*) = L_{N',N}^{y_h} \tag{3.4}$$

where λ_h is defined by $h' = \lambda_h(p)h$, which is obtained from the leading order in h of (3.1). These exponents (y_p and y_h) and the surface one (y_{hs}) are not uniquely determined. We obtain slightly different values from the mappings. Although only two maps are defined in (3.1), a third one involving the clusters with N' and N'' can be defined. It has the same fixed point and the same $2-y_h-y_{hs}$ exponent as the maps defined above.

Note that the infinite cluster is characterized by two correlation lengths [4], one along the main diagonal $\xi_\parallel \sim (p-p_c)^{-\nu_\parallel}$ and another perpendicular to the main diagonal $\xi_\perp \sim (p-p_c)^{-\nu_\perp}$. Only $\xi_\parallel$ rescales like a simple length under the cluster transformations considered here. So we have that $\nu_\parallel = 1/y_p$ [9].

In table 1 we present the fixed points and the exponents obtained by fixing two clusters with $N=78$ and $N'=66$ sites and allowing the third one to change from $N''=1$ up to $N''=55$. Only the exponents obtained by the maps defined in (3.1) are shown. It is worth mentioning that the fixed points for the last clusters are already very near to the best value of the percolation threshold obtained from series expansions $p_c = 0.705\,489 \pm 4$ [10]. We can also observe for the largest clusters that the exponents obtained from different maps with N, N' and N'' fixed have very close values.

Table 1. Results of the mean-field renormalization group (an approach to bulk and surface properties) for the clusters with $N=78$ and $N'=66$ sites fixed. The third cluster has N'' sites.

N''	p_c	$y_p^{N,N'}$	$y_p^{N,N''}$	$y_h^{N,N'}$	$y_h^{N,N''}$	$y_{hs}^{N,N'}$	$y_{hs}^{N,N''}$
1	0.709 931	0.714 12	0.872 94	1.668 99	1.727 80	0.610 86	0.552 05
3	0.703 225	0.709 02	0.807 57	1.674 23	1.707 73	0.620 70	0.587 19
6	0.703 671	0.708 02	0.776 49	1.675 24	1.698 04	0.622 59	0.599 79
10	0.703 955	0.707 39	0.757 39	1.675 89	1.692 17	0.623 79	0.607 51
15	0.704 143	0.706 97	0.744 05	1.676 31	1.688 16	0.624 58	0.612 72
21	0.704 274	0.706 67	0.734 04	1.676 61	1.685 22	0.625 13	0.616 51
28	0.704 372	0.706 45	0.726 18	1.676 83	1.682 96	0.625 54	0.619 41
36	0.704 447	0.706 28	0.719 78	1.677 00	1.681 15	0.625 86	0.621 71
45	0.704 509	0.706 15	0.714 43	1.677 14	1.679 66	0.626 11	0.623 59
55	0.704 559	0.706 03	0.709 88	1.677 25	1.678 41	0.626 33	0.625 17

3.2. Extrapolation of the percolation threshold

In order to extrapolate the sequence of fixed points and obtain an estimate of the critical threshold p_c, we consider that $\xi_\parallel(p_c) \sim L_{N''}$ where $L_{N''}$ is the linear dimension of a finite cluster. Since N and N' are fixed for all considered clusters, we have taken $L_{N''} = \sqrt{N''}$, the linear dimension of the third changing cluster. This is natural because the other linear dimensions are equal for all three cluster series and they enter the expression for $\xi_\parallel$ as a constant. A rather good least-squares fit of $\ln(p_c - p^*) \times (-1/\nu_\parallel) \ln L$ with the assumed value $\nu_\parallel = 1.733$ is obtained. Table 2 shows the variation of the central estimate of the critical threshold (p_c) with the number of data points

Table 2. Estimates of p_c based on the 10 points obtained from the three-cluster MFRG with the clusters of $N = 78$ and $N' = 66$ sites fixed. Shown is the variation of the central estimate with the number (M) of considered points. p_c has been obtained by fitting with the variable $\sqrt{N''}$. The best series estimate is $p_c = 0\,705\,489 \pm 4$ [10].

M	$p_c(M)$
10	0 706 32
9	0 705 60
8	0 705 56
7	0 705 52
6	0 705 48
5	0 705 45

(M) considered in the case where $N = 78$ and $N' = 66$ are fixed. It shows the extrapolation with $M = 10$ points. However the first points must be worse than the last ones because of the small sizes of the first clusters. So, we consider $M = 9$ points by ignoring the first one, which corresponds to the fixed point of the ($N = 78$, $N' = 66$, $N'' = 1$) clusters. When the second fixed point is also ignored, we have $M = 8$ and so on. We can conclude that $p_c = 0.7055 \pm 1$ (the error is in the last digit), a value which is in very good agreement with the best evaluation of the series. Moreover, the computational cost of the present work is lower when compared with series work. We have evaluated the estimates of p_c with other fixed clusters ($N'' = 1$, $N' = 3, 6, \ldots, 78$). Except for the first case ($p_c = 0.7060 \pm 5$), we have obtained the same estimate $p_c = 0.7055$, with the error changing from ± 3 for the first clusters to ± 2 for the last ones.

Another way to do the extrapolation for p_c can be obtained by considering that $\xi_\parallel \sim L_{\parallel,N''}$ instead of $\xi_\parallel \sim L_{N''}$. Here $L_{\parallel,N''}$ is an effective length, defined in [11] as an average projection along the parallel direction of the vectors beginning at the origin site and ending in each site of the cluster. A good fit is obtained with $\ln(p_c - p^*) \times (-1/\nu_\parallel) L_{\parallel,N''}$. The estimates of the percolation threshold are also very good and agree with the values obtained previously. This indicates that the two scaling factors are similar for the asymmetric clusters, in agreement with Redner [9].

It is important to mention that the usual two clusters MFRG application to the two-dimensional directed percolation [5] has been given an estimate of p_c slightly out from the best value. This has occurred because the boundary effective field b has not been scaled as a surface one. Once we correct this point, we obtain good estimates of p_c.

3.3. Extrapolation of the exponents

In order to estimate y_p by extrapolation processes, we follow an approach similar to that used by Reynolds *et al* [12]. Let us consider the eigenvalue λ_p, calculated from the three-cluster method. A correction term $A(L, \sqrt{N''})$, depending not only on the scaling factor between the two clusters but also on the linear dimension of the third cluster entering the renormalization group mapping, can be defined by the following relation:

$$\lambda_p^{\text{true}}(L) = A(L, \sqrt{N''}) \lambda_p^{\text{calc}}(L, \sqrt{N''}). \tag{3.5}$$

Here $L = \sqrt{N/N'}$. We expect that λ_p^{calc} approaches the true value when $\sqrt{N''} \to \infty$ and $L \to \infty$ or $L \to 1$. Thus we must do two extrapolations: one involving the limit of $\sqrt{N''}$

and the other implying $L \to \infty$ or $L \to 1$. We suppose that for N'' big enough the correction term behaves as

$$A(L, \sqrt{N''}) \sim A(L)\left[1 + \frac{c}{\sqrt{N''}}\right] \tag{3.6}$$

where c is a constant and $A(L)$ is the correction term involving only the scaling factor.

Let us discuss the limit $L \to 1$. In this limit we can write that $L \sim 1 + \delta$, where δ is a small deviation of L from its asymptotic value. Using that $\lambda_p = L^{y_p}$ we obtain from (3.5) and (3.6)

$$y_p^{\text{true}} = \frac{A(L) - 1}{\delta} + y_p^{\text{calc}}(L, \sqrt{N''})A(L) + \frac{a}{\sqrt{N''}}. \tag{3.7}$$

This equation suggests that the sequence $y_p^{\text{calc}}(L, \sqrt{N''})$ should be first extrapolated against the variable $1/\sqrt{N''}$. In the asymptotic region we have a straight line with an intercept given by

$$y_p'(L) = \frac{y_p^{\text{true}}}{A(L)} - \frac{A(L) - 1}{A(L)\delta} \tag{3.8}$$

The next step is to consider the obtained sequence $y_p'(L)$. Since we expect that y_p' approaches the true exponent in the limit $\delta \to 0$, $A(L)$ cannot depend on a linear term in δ. Near $L = 1$ we must have that $A(L) \sim 1 - \delta^2 K$, where K is a positive constant. Then from (3.8) we can write

$$y_p' = y_p^{\text{true}} + \delta K \tag{3.9}$$

suggesting that the extrapolation must be made in the variable $\delta = L - 1$. The intercept is equal to y_p^{true} in the asymptotic region. A similar discussion can be made for the limit $L \to \infty$.

In table 3 we show the estimates of y_p' with the number M of considered points when the clusters of $N = 78$ and $N' = 66$ are fixed and the third one varies from 1 site to 55 sites. The y_p^{calc} sequence plotted against $1/\sqrt{N''}$ is in a very good straight line. We have also changed the exponent of the N'' in order to check the 1/2 value. The estimate of y_p' is essentially independent of the exponent for values near 1/2. Moreover, the best fit is obtained with a value very close to 1/2. The estimates of y_p^{true} with M

Table 3. Estimates of the partial quantities (y_p', y_h' and y_{hs}') related to the critical exponents obtained from the three-cluster MFRG with the clusters of $N = 78$ and $N' = 66$ sites fixed. Shown is the variation of the central estimate with the number (M) of considered points. The fitting was made with the variable $1/\sqrt{N''}$.

M	y_p'	y_h'	y_{hs}'
10	0 704 64	1 678 68	0.629 01
9	0 705 17	1.678 13	0.627 96
8	0.705 07	1.678 23	0.628 14
7	0.705 04	1 678 26	0.628 19
6	0 705 02	1 678 28	0 628 23
5	0.705 00	1 678 29	0.628 25
4	0.704 99	1 678 31	0.628 28

Table 4. Estimates of y_p, y_h and y_{hs} based on the 11 points obtained from the three-cluster MFRG in the limit $L \to 1$. Shown is the variation of the central estimate with the number (M) of considered points. The expected value for y_o is 0.5767^{+2}_{-4} [10].

M	y_p	y_h	y_{hs}
11	0.671 93	1.665 53	0.648 47
10	0.654 25	1.661 46	0.649 15
9	0.646 17	1.659 34	0.651 44
8	0.640 74	1.658 29	0.652 42
7	0.636 83	1.657 72	0.653 07
6	0.633 92	1.657 27	0.653 37
5	0.631 58	1.656 95	0.653 58
4	0.629 65	1.656 70	0.653 73

are shown in table 4. In this case we consider the $L \to 1$ limit. We obtain that $\nu_\parallel = 1.58 \pm 1$, a value which is 10% out from the expected one ($\nu_\parallel = 1.7339 \pm 3$) [10]. We have also considered the $L \to \infty$ limit by fixing the $N' = 10$ cluster, and allowing the others to go to infinity with the procedures described above. We have obtained that $\nu_\parallel = 1.53 \pm 2$.

The $\nu_\parallel$ exponent can also be obtained by considering that the scaling factor $L_\parallel$ between two clusters is given by the ratio of the effective lengths ($L_{\parallel,N'}/L_{\parallel,N}$) [11] as discussed before. Again we have an exponent depending on the scaling factor and on the linear dimension of the third cluster. Thus we can use the same analysis. We obtain the same $\nu_\parallel$ as before in the two limits ($L_\parallel \to 1$, $L_\parallel \to \infty$) with the same fitting quality. It means that for these asymmetric clusters the effective length variable is equivalent to the $\sqrt{N}$ length variable as argued by Redner [9].

Extrapolations of y_h and y_{hs} follow precisely the same arguments as for y_p. In table 3 is shown the variation of the central estimates for y'_h and y'_{hs} with the number M of considered points when clusters of $N = 78$ and $N' = 66$ sites are maintained fixed and the third one is varying from 1 up to 55 sites. The case $L \to 1$ is presented in table 4. There we have the variation of the central estimates of y_h and y_{hs} with the number of points considered. We conclude that $y_h = 1.657 \pm 1$ and $y_{hs} = 0.653 \pm 1$. This is the first evaluation of y_{hs} for the directed percolation problem. The value of y_h indicates that the directed exponents do not obey the usual scaling law $y_h = \Delta y_p = 1.47$ [13].

4. Summary

The mean-field renormalization group approach to bulk and surface properties has been applied to the two-dimensional site directed percolation problem. We have used mean-field asymmetric clusters with 1 up to 78 sites. We have obtained a very good estimate of $p_c = 0.7055 \pm 1$. The obtained $\nu_\parallel$ exponent is 10% smaller than the expected value. We believe that by considering a few more clusters this last result will be improved. We have evaluated also the exponent y_h and found for the first time the value for the surface exponent $y_{hs} = 0.653 \pm 1$. Several extrapolation schemes have discussed in order to obtain the three-cluster MFRG results in a systematic way. Finally, it is worth mentioning that in the three-cluster MFRG the incorrect assumption of the two-cluster MFRC that the boundary effective field scales as the order parameter has been corrected. This is essential in the study of the convergence of the MFRG results.

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